On the theoretical side I have shown that J. J. Thomson's formula is confirmed by the present method of analysis, and that the Lorentz-Einstein formula for a "contracted" electron requires to be modified in the manner stated.

If it be argued, as may fairly be done, that the experiments do not conflict with the Lorentz-Einstein formula, we require to give up the view that the electron is contracted in the proportion they assume. But it is fairly certain that there is an electron of form intermediate between Abraham's sphere and Lorentz' spheroid which would have a transverse inertia correctly given by $m_0 (1-k^2)^{-\frac{1}{2}}$. This, however, implies giving up the correspondence assumed in the relativity theory, and we might choose between this and the alternative view (by no means unreasonable) that electrons as obtained in experiments possess ordinary as well as electric inertia.

It appears that still further experiments are required, and it would be helpful if results were represented in a form that could be used to test any formula and not any particular one. The subject is still too uncertain to warrant the view that any special formula represents the end of the matter.

On the Mode of Approach to Zero of the Coefficients of a Fourier Series.

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§ 1. In the present communication I give a number of results on the mode of approach to zero of the coefficients of a Fourier series, to which I have already made allusion in my paper on "The Order of Magnitude of the Coefficients of a Fourier Series." These results are not merely curious, they have a real importance, and give one an insight into the nature of these series, which cannot easily be gained without them. Indeed, while the earlier paper leads, as I showed in a subsequent communication, to the discovery of classes of derived series of Fourier series, which, although not themselves Fourier series, none the less converge, and are utilisable in a similar manner, and is therefore in a certain sense of practical interest, the present paper does something towards the elucidation of the general theory of the convergence of Fourier series themselves, as well as of their derived series.

It will be sufficient to give a single instance. I have recently shown that the well-known test of Dirichlet for the convergence of a Fourier series admits of a remarkable generalisation. It follows from Theorem 1, given below (§ 7), that the convergence secured by that test and by its generalisation alike possess what may be called greater strength than the rival tests of Dini and de la Vallée Poussin.

§ 2. In the course of the proofs of the auxiliary theorems A1, ..., we shall require the following Lemmas:

LEMMA 1.—*The expression* $(x+1)^p-x^p$ *is* $\lesseqgtr 0$ *according as* $p \lesseqgtr 0$, *and is monotone descending or ascending with respect to* x *according as* $p \lessgtr 1$.

LEMMA 2.—*For all values of* $p>1$, *the expression* $(x+1)^p-2x^p+(x-1)^p$ *is positive, and it decreases or increases with respect to* x *according as* $p \lessgtr 2$.

LEMMA 3.—*If* $p_1>p_2>p_3>\ldots>p_n>0$, *then*

$$|p_1u_1+p_2u_2+\ldots+p_nu_n| \leq p_1 \times \textit{max. value of } (u_1+u_2+\ldots+u_r), \textit{ for } 1 \leq r \leq n.$$

Also if $0<p_1<p_2<p_3<\ldots<p_n$, *then*

$$|p_1u_1+p_2u_2+\ldots+p_nu_n| \leq 2p_n \times \textit{max. value of } (u_1+u_2+\ldots+u_r), \textit{ for } 1 \leq r \leq n.$$

LEMMA 4.—*If* $S_r = \frac{1}{2}+\sum_{s=1}^{r} \cos sx$, *or if* $S_r = -\frac{1}{2}\cot\frac{1}{2}x+\sum_{s=1}^{r} \sin sx$, *then*

$$|S_r| \leq \tfrac{1}{2}\left|\operatorname{cosec}\tfrac{1}{2}x\right|;$$

and if $T_r = \sum_{s=1}^{r} S_s$, *then* $\quad |T_r| \leq \frac{1}{2}\operatorname{cosec}^2\frac{1}{2}x$.

§ 3. THEOREM A1.—*If* $0<p<1$, *and* $0<v\leq e$, *where* $e\leq\pi$, *then either of the expressions*

$$(nv)^p\left\{\sum_{r=2}^{n-1}\left(\genfrac{}{}{0pt}{}{\sin}{\cos} rv-\genfrac{}{}{0pt}{}{\sin}{\cos}(r-1)v\right)(1-r/n)^p-(1-1/n)^p\genfrac{}{}{0pt}{}{\sin}{\cos} v\right\}$$

is a bounded function of (v, n),

The argument being the same in both cases, we only treat the sine expression, which we shall denote by S.

Let us consider separately the values of $nv\leq B$ and those $>B$, where B is any fixed positive quantity.

In the former case,

$$|S| = \left|v^p\sum_{r=1}^{n-1}\sin rv\left[(n-r)^p-(n-r-1)^p\right]\right| \leq v^p\sum_{r=1}^{n-1}\left[(n-r)^p-(n-r-1)^p\right] \leq B^p,$$

so that S is bounded.

In the latter case, $v>B/n$, and we have

$$S = v^p\sum_{r=1}^{n-1}\sin rv\left[(n-r)^p-(n-r-1)^p\right] = v^p\sum_{r=1}^{n-1}\sin(n-r)v\left[r^p-(r-1)^p\right]$$

$$= v^p\sum_{r=1}^{n-1}(\sin nv\cos rv-\cos nv\sin rv)\left[r^p-(r-1)^p\right].$$

Hence
$$|S| \leq |L_p| + |M_p| \tag{1}$$

where
$$L_p = v^p \sum_{r=1}^{n-1} \cos rv\,[r^p - (r-1)^p], \tag{2}$$

and M_p only differs from L_p in having sines in place of the cosines.

Let us divide the interval $(0, nv)$ into n equal parts; each of these is, by hypothesis, less than e, and the point B will lie in one of them, or coincide with one of the points of division. Denoting this part as the $(s+1)$-th, we have therefore,

$$sv \leq B < (s+1)\,v < B + e. \tag{3}$$

Divide the summation L_p into two parts at the s-th term. Writing $L_{p,1}$ and $L_{p,2}$ for the two parts, we have

$$|L_{p,1}| = \left|v^p \sum_{r=1}^{s-1} \cos rv\,[r^p - (r-1)^p]\right| \leq v^p \sum_{s=1}^{s-1} [r^p - (r-1)^p] \leq v^p s^p \leq B^p. \tag{4}$$

Also, using our Lemmas 1, 3 and 4,

$$|L_{p,2}| \leq v^p\,[(s+1)^p - s^p] \times \text{max. value of } \sum_{r=s+1}^{k} \cos rv, \qquad (s+1 \leq k \leq n-1),$$

$$\leq v^p\,[(s+1)^p - s^p] \operatorname{cosec} \tfrac{1}{2}v \leq 2B^{p-1} p\,(\tfrac{1}{2}e \operatorname{cosec} \tfrac{1}{2}e), \tag{5}$$

since $\frac{1}{2}v \operatorname{cosec} \frac{1}{2}v$ is a monotone increasing function of v from 0 to π.

Thus, $L_{p,1}$ and $L_{p,2}$ being bounded, L_p is bounded, and similarly M_p is bounded, therefore, by (1), S is bounded, which proves the theorem.

§ 4. THEOREM A2.—*If* $1 < p$ *and* $0 < v \leq e$, *where* $e \leq \pi$, *then either of the expressions*

$$nv \left\{ \sum_{r=2}^{n-1} \left(\genfrac{}{}{0pt}{}{\sin}{\cos} rv - \genfrac{}{}{0pt}{}{\sin}{\cos} (r-1)\,v \right) (1 - r/n)^p - (1 - 1/n) \genfrac{}{}{0pt}{}{\sin}{\cos} v \right\}$$

is a bounded function of (v, n).

This theorem may be most easily proved by the same method as that used in the preceding article, bearing in mind that, since p is now greater than unity, $\{r^p - (r-1)^p\}$ is positive and monotone increasing. We prefer, however, to give the following method, as it involves certain expressions required in the sequel. The equations (1)—(4) of the preceding article still hold. We then proceed as follows:—

$$L_{p,2} = v^p \sum_{r=s+1}^{n-1} (S_r - S_{r-1})\,[r^p - (r-1)^p]$$

$$= v^p \Big\{ S_s\,[(s+1)^p - s^p] - \sum_{r=s+1}^{n-1} S_r\,[(r+1)^p + (r-1)^p - 2r^p]$$

$$+ S_{n-1}\,[(n-1)^p - (n-2)^p] \Big\}, \tag{6}$$

where
$$S_r = \tfrac{1}{2} + \cos v + \cos 2v + \ldots + \cos rv.$$

By Lemma 2, the expression $[(r+1)^p+(r-1)^p-2r^p]$ is positive and monotone decreasing, if $1<p<2$. Hence, by Lemmas 3 and 4, the corresponding summation which occurs in the expression for $L_{p,2}$ is numerically less than

$$\operatorname{cosec}^2 \tfrac{1}{2}v\,[(s+1)^p+(s-1)^p-2s^p]$$
$$= s^{p-2}p\,(p-1)\operatorname{cosec}^2 v\left\{\frac{(s+1)^p+(s-1)^p-2s^p}{p\,(p-1)\,s^{p-2}}\right\} \leq s^{p-2}p\,(p-1)\operatorname{cosec}^2 v.$$

This has to be multiplied by v^p, and is therefore evidently bounded, since, by (3) of the preceding article, $(vs)^{p-2}$ is bounded.

Also, as in the discussion of (5) of the preceding article,

$$v^p S_s\,[(s+1)^p-s^p] \leq 2B^{p-1}p\,(\tfrac{1}{2}e \operatorname{cosec} \tfrac{1}{2}e),$$

so that this is also bounded. Thus

$$L = 0\,(1)+v^p S_{n-1}\,[(n-1)^p-(n-2)^p], \qquad (1<p<2), \tag{7}$$

which is the expression above referred to as required in the sequel and leads to our present theorem, since by Lemma 4, vS_{n-1} is bounded, so that

$$L_p = 0\,(nv)^{p-1}.$$

A similar result holding for M, it follows, as in the preceding article, that $(nv)^{-p+1}S$, which is the expression given in the enunciation of the present theorem, is bounded. This proves the theorem, when $1<p<2$.

If $2 \leq p$, we still have (6); but now $\{(r+1)^p+(r-1)^p-3r^p\}$ is monotone increasing; the term with the summation on the right of (6) is therefore numerically less than

$$v^p\, 2 \operatorname{cosec}^2 \tfrac{1}{2}v\,[(n+1)^p+(n-1)^p-2n^p] = 0\,(nv)^{p-2},$$

since, writing x in the place of n^{-1}, we have by indeterminate forms,

$$\operatorname*{Lt}_{n\to\infty} n^{-p+2}\,[(n+1)^p+(n-1)^p-2n^p]$$
$$= \operatorname*{Lt}_{x\to 0} x^{-2}\,[(1+x)^p+(1-x)^p-2] = p\,(p-1).$$

Instead of (7), we have now

$$L_p = 0\,(nv)^{p-2}+v^p S_{n-1}\,[(n-1)^p-(n-2)^p)], \qquad (2 \leq p), \tag{7'}$$

whence our result follows as before.

§ 5. THEOREM B1.—*If* $0<p<2$, *and* $0<v \leq e$, *where* $e \leq \pi$, *then either of the expressions*

$$(nv)^p \sum_{r=1}^{n-1} n^{-1} \left\{ r \begin{matrix}\sin\\ \cos\end{matrix}\, rv-(r-1) \begin{matrix}\sin\\ \cos\end{matrix}\,(r-1)\,v \right\} (1-r/n)^p$$

is a bounded function of (v, n).

It is only necessary to discuss the sine expression.

If $nv < \mathrm{B}$, the expression is clearly bounded. For, denoting it by T, we have, since $(1-r/n)^p$ is monotone descending,

$$|\mathrm{T}| \leq \frac{1}{n} (1-1/n)^p \times \text{max. value of} \sum_{r=1}^{k} [r \sin rv - (r-1)\sin(r-1)v]$$
$$\leq (1+1/n)(1-1/n)^p.$$

If $\mathrm{B} \leq nv$, we have

$$\mathrm{T} = v^p \sum_{r=1}^{n-1} \frac{r}{n} [(n-r)^p - (n-r-1)^p] \sin rv$$

$$= v^p \sum_{r=1}^{n-1} (1-r/n)[r^p - (r-1)^p] \sin(n-r)v = \mathrm{P}_1 \sin nv - \mathrm{P}_2 \cos nv, \qquad (1)$$

where

$$\mathrm{P}_1 = v^p \sum_{r=1}^{n-1} (1-r/n)[r^p - (r-1)^p] \cos rv$$

$$= v^p \sum_{r=1}^{n-1} \{[r^p - (r-1)^p] - n^{-1}[(r^{p+1} - (r-1)^{p+1} + (r-1)^p]\} \cos rv$$

$$= \mathrm{L}_p - (nv)^{-1} \mathrm{L}_{p+1} + n^{-1} \mathrm{M}_p, \qquad (2)$$

where L_p has the same meaning as in the preceding articles, L_{p+1} being the same, with $(p+1)$ in place of p, and

$$\mathrm{M}_p = v^p \sum_{r=1}^{n-1} (r-1)^p \cos rv. \qquad (3)$$

If $p < 1$, L_p is bounded by Theorem A1, and by Theorem A2 L_{p+1} is of the order $0\,(nv)^p$. Also since $(r-1)^p$ is positive and monotone increasing,

$$|n^{-1}\mathrm{M}_p| \leq n^{-1} v^p \, 2(n-2)^p \operatorname{cosec} \tfrac{1}{2}v \quad (nv)^{p-1} \, 4(\tfrac{1}{2}v \operatorname{cosec} \tfrac{1}{2}v) \leq \mathrm{B}^{p-1} \, 2e \operatorname{cosec} \tfrac{1}{2}e.$$

Thus P_1 is bounded. Similarly P_2 is bounded, and therefore, by (1), the required result follows, when $0 < p < 1$.

If $1 \leq p < 2$, we have, using the expressions (7) and (7′), found in § 4,

$$\mathrm{L}_p = 0\,(1) + v^p \mathrm{S}_{n-1} [(n-1)^p - (n-2)^p],$$

$$(nv)^{-1} \mathrm{L}_{p+1} = 0\,(nv)^{p-2} + v^p \mathrm{S}_{n-1} n^{-1} [(n-1)^{p+1} - (n-2)^{p+1}].$$

Also

$$\mathrm{M}_p = v^p \sum_{r=1}^{n-1} (\mathrm{S}_r - \mathrm{S}_{r-1})(r-1)^p = v^p \sum_{r=1}^{n-2} \mathrm{S}_r \{(r-1)^p - r^p\} + v^p \mathrm{S}_{n-1} (n-2)^p.$$

Since $[r^p - (r-1)^p]$ is positive and monotone ascending, the summation here is numerically less than

$$2 \operatorname{cosec}^2 \tfrac{1}{2}v [n^p - (n-1)^p] = 8v^{-2} (\tfrac{1}{2}e \operatorname{cosec} \tfrac{1}{2}e)^2 p n^{p-1} \left(\frac{1-(1-1/n)^p}{p/n}\right),$$

where the last factor on the right → 1, when $n \to \infty$. Since then

$$(nv)^{p-2} < \mathrm{B}^{p-2}, \qquad n^{-1}\mathrm{M}_p = 0\,(1) + v^p \mathrm{S}_{n-1} n^{-1} (n-2)^p.$$

Hence

$$P_1 = L_p-(nv)^{-1}L_{p+1}+n^{-1}M_p = O(1)+v^p S_{n-1}n^{-1}[(n-1)^p-(n-2)^p]$$

$$= O(1)+v\,(vS_{n-1})\,(nv)^{p-2}p\left(\frac{1-[1-1/(n-1)]^p}{p/(n-1)}\right) = O(1).$$

Thus P_1 is again bounded, whence as before the required result follows, when $1 \leq p < 2$. This completes the proof of the theorem.

§ 6. The same method, using only the formula (7′) and not (7), gives us the theorem :—

THEOREM B2.—If $2 \leq p$, and $0 \leq v \leq e$, where $e \leq \pi$, then either of the expressions

$$(nv)^{-2}\sum_{r=1}^{n-1} n^{-1}\left\{r \genfrac{}{}{0pt}{}{\sin}{\cos} rv-(r-1)\genfrac{}{}{0pt}{}{\sin}{\cos}(r-1)\,v\right\}(1-r/n)^p$$

is bounded.

We foresee now the general theorem :—

THEOREM.—*If* $0<p<q+1$, *and* $0<v\leq e$, *where* $e \leq \pi$, *then either of the expressions*

$$(nv)^p\sum_{r=1}^{n-1} n^{-q}\left\{r^q \genfrac{}{}{0pt}{}{\sin}{\cos} rv-(r-1)\genfrac{}{}{0pt}{}{\sin}{\cos}(r-1)\,v\right\}(1-r/n)^p$$

is bounded. And if $q+1 \leq p$, *the same is true when we change the index* p *in* $(nv)^p$ *into* $q+1$.

The proof of this theorem may be left to the student.

§ 7. We now pass to the main theorems of the paper.

THEOREM 1.—*If* a_n *is the typical coefficient of the Fourier cosine series of an even function which is such that*

(i) $x^{-1}\int_0^x \left| d\{xf(x)\} \right|$ *is a bounded function of* x *in the whole interval of periodicity, and*

(ii) $x^{-1}\int_0^x f(x)\,dx$ *has a unique finite limit* β *as* $x \to 0$,

then $(Ck)(na_n) \to 0$, *when* $n \to \infty$, $(0<k<1)$.

We have

$$\tfrac{1}{2}\pi\,(Ck)\,(na_n) = \int_0^\pi \sum_{r=1}^{n-1}[r\cos rv-(r-1)\cos(r-1)\,v]\,(1-r/n)^k f(v)\,dv$$

$$= \int_0^\pi \psi'_n(v) f(v)\,dv$$

where
$$\psi_n(v) = \sum_{r=1}^{n-1}[\sin rv-\sin(r-1)\,v]\,(1-r/n)^k.$$

Thus, by Theorem A1, $(nv)^k\psi_n(v)$ is bounded for all positive values of v.

Let us consider first the part of the integral due to the interval $(B/n, \pi)$, where B is a large positive quantity, independent of n, and let us only consider values of n greater than B/π. Let the integral in this interval be denoted by I_1, and in the remaining interval by I_2. We have then

$$I_1 = \int_{B/n}^{\pi} \psi'_n(v) f(v)\, dv = \int_{B/n}^{\pi} v^{-1} \psi'_n(v)\, v f(v)\, dv.$$

Now in the interval $(-\pi, \pi)$ the expression $vf(v)$ is a function of bounded variation, for its total variation is $2\int_0^{\pi} \left| d[vf(v)] \right|$, which by (i) is finite. Let us denote this function of bounded variation by $g(v)$. Then by (i), $\left| x^{-1} g(x) \right| \leq x^{-1} \int_0^x \left| dg(x) \right| < A'$, where A' is independent of B and n, we may therefore write

$$I_1 = \int_{B/n}^{\pi} \phi'_n(v)\, g(v)\, dv = \Big[\phi_n(v)\, g(v) \Big]_{B/n}^{\pi} - \int_{B/n}^{\pi} \phi_n(v)\, dg(v),$$

where
$$\phi_n(v) \leq v^{-1} \psi_n(v) + \int_x^{\pi} v^{-2} \psi_n(v)\, dv.$$

It follows at once that $(nv)^k v\, \phi_n(v)$ is a bounded function, say numerically less than A, where A is independent of B and n.

We have, therefore, denoting $\int_0^v \left| dg(v) \right|$ by $G(v)$,

$$\left| I_1 \right| \leq \left| \Big[\phi_n(v) g(v) \Big]_{B/n}^{\pi} \right| + \int_{B/n}^{\pi} \left| \phi_n(v) \right| dG(v)$$
$$\leq AA' \left\{ (n\pi)^{-k} + B^{-k} \right\} + \int_{B/n}^{\pi} A\, (nv)^{-k} v^{-1}\, dG(v).$$

Using integration by parts for the evaluation of the last integral, this gives

$$\left| I_1 \right| \leq AA' \left\{ (n\pi)^{-k} + B^{-k} \right\} + A \Big[(nv)^{-k} v^{-1} G(v) \Big]_{B/n}^{\pi}$$
$$+ A(k+1) \int_{B/n}^{\pi} (nv)^{-k} v^{-2}\, G(v)\, dv$$
$$\leq 2AA' \left[(n\pi)^{-k} + B^{-k} \right]$$
$$+ AA'(1 + 1/k) \int_{B/n}^{\pi} d(nv)^{-k} \leq AA'(3 + 1/k) \left[(n\pi)^{-k} + B^{-k} \right].$$

Hence
$$\operatorname*{Lt}_{n \to \infty} \left| I_1 \right| \leq AA'(3 + 1/k)\, B^{-k},$$

which is as small as we please, B being chosen sufficiently large.

Next, changing the variable to $t = nv$, and writing $\chi_n(t) = n^{-1} \psi_n'(t/n)$, we have

$$I_2 = \int_0^{B/n} \psi'_n(v) f(v)\, dv = \int_0^{B} \chi_n(t) f(t/n)\, dt$$
$$= \Big[n \chi_n(t) F(t/n) \Big]_0^{B} - \int_0^{B} n t^{-1} F(t/n)\, t \frac{d}{dt} \chi_n(t)\, dt.$$

Now $t\frac{d}{dt}\chi_n(t)$ is easily seen to be a bounded function of (t, n)*, and, as $n \to \infty$, it approaches $t\frac{d}{dt}\lambda(t)$, where

$$\underset{n\to\infty}{\mathrm{Lt}}\ \chi_n(t) = \lambda(t) = \int_0^1 (1-u)^k d(u\cos tu) = k\int_0^1 (1-u)^{k-1} u\cos tu\, du$$

$$= k\frac{d}{dt}\int_0^1 (1-u)^{k-1}\sin tu\, du = k\Gamma(k)\frac{d}{dt}\left[t^{-k}\mathrm{C}_{k+1}(t)\right].\dagger$$

Again, $nt^{-1}\mathrm{F}(t/n) \to \beta$, as $n \to \infty$, by the hypothesis (i); and it is a bounded function of (n, t). Thus, both factors of the integrand of the integral in the expression last given for I_2 being bounded, we may,‡ when $n \to \infty$, integrate term by term, and get

$$\underset{n\to\infty}{\mathrm{Lt}}\ \mathrm{I}_2 = \Big[\beta t\lambda(t)\Big]_0^{\mathrm{B}} - \beta\int_0^{\mathrm{B}} t\frac{d}{dt}\lambda(t)\,dt = \beta\int_0^{\mathrm{B}}\lambda(t)\,dt$$

$$= \beta\Gamma(k+1)\Big[t^{-k}\mathrm{C}_{k+1}(t)\Big]_0^{\mathrm{B}} = \beta\Gamma(k+1)\,\mathrm{B}^{-k}\mathrm{C}_{k+1}(\mathrm{B}),$$

which is of the order B^{-k}, and is therefore as small as we please, B being chosen sufficiently large.

This proves the theorem.

§ 8. The companion theorem is as follows :—

THEOREM 2.—*If* b_n *is the typical coefficient of the Fourier sine series of an odd function, which is such that*

(i) $x^{-1}\int_0^x \big| d[xf(x)] \big|$ *is a bounded function of* x *in the whole interval of periodicity, and*

(ii) $x^{-1}\int_0^x f(x)\,dx$ *has a unique finite limit* β *as* $x \to 0$,

then $(\mathrm{C}k)(nb_n) \to 2\beta/\pi$, *when* $n \to \infty$.

The proof is the same as that of the preceding article, sines taking the

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* In fact

$$\left| t\frac{d}{dt}\chi_n(t) \right| = \left| \sum_{r=1}^{n-1} \frac{r^2}{n^2} t\sin\frac{r}{n}t\left\{(1-r/n)^k - [1-(r+1)/n]^k\right\} \right|$$

$$\leq \mathrm{B}\sum_{r=1}^{n-1}\left([1-r/n]^k - [1-(r+1)/n]^k\right) < \mathrm{B}.$$

† W. H. Young, "On Infinite Integrals involving a Generalisation of the Sine and Cosine Functions," 'Quart. Journ.,' vol. 43, p. 163 (1912).

See, for instance, my paper on "Successions of Integrals and Fourier Series," 'Proc. L.M.S.,' Ser. 2, vol. 11, p. 78, Cor. 1 (1912).

place of cosines. The only essential difference is in the expression for the function $\lambda(t)$. We now have

$$\lambda(t) = \int_0^1 (1-u)^k \, d(u \sin tu) = k \int_0^1 (1-u)^{k-1} u \sin tu \, dt$$
$$= -k \frac{d}{dt} \int_0^1 (1-u)^{k-1} \cos tu \, dt = -k \Gamma(k) \frac{d}{dt} [t^{-k} C_k(t)].$$

Whence

$$\operatorname*{Lt}_{n\to\infty} I_2 = \beta \int_0^B \lambda(t) \, dt = \beta \Gamma(k+1)[1/\Gamma(k+1) - B^{-k} C_k(B)],$$

which, since B is as large as we please, so that the part of this expression involving B may be neglected, gives

$$\operatorname*{Lt}_{n\to\infty} I_2 = \beta.$$

This proves the theorem.

§ 9. THEOREM 3.—*If a_n is the typical coefficient of the Fourier cosine series of an even function which is such that*

(i) $x^{-2} \int_0^x \left| d[xf(x)] \right|$ *is a bounded function of x in the whole interval of periodicity, and*

(ii) $2x^{-2} \int_0^x f(v) \, dv$ *has a unique limit β as $x \to 0$,*

then $(C, 1+k)(n^2 a_n) \to 2\beta/\pi$, as $n \to \infty$, $(0 < k < 1)$.

We have

$$\frac{\pi}{2}(C, 1+k)(n^2 a_n) = \int_0^\pi \sum_{r=1}^{n-1} \left[r^2 \cos rv - (r-1)^2 \cos (r-1) v \right] (1 - r/n)^{1+k} f(v) dv$$
$$= \int_0^\pi \psi'_n(v) f(v) \, dv,$$

where
$$\psi_n(v) = \sum_{r=1}^{n-1} \left[r \sin rv - (r-1) \sin (r-1) v \right] (1 - r/n)^{1+k}.$$

By Theorem B1, $n^k v^{1+k} \psi_n(v)$ is bounded for all positive values of v as $n \to \infty$.

Writing, as before, I_1 and I_2 for the two parts of the integral due to the intervals $(B/n, \pi)$ and $(0, B/n)$, we find, precisely as in the preceding article, that, as $n \to \infty$, I_1 approaches a limit or limits as small as we please, B being conveniently large.

The treatment of I_2 is the same as in the preceding article; we only have to notice that $nt^{-2} F(t/n)$ is now bounded, and that $\chi_n(t)$ now stands for

$n^{-2}\psi'_n(t/n)$, $\psi_n(t)$ having the meaning attributed to it in the present article. The limiting function $\lambda(t)$ is, therefore,

$$\lambda(t) = \operatorname*{Lt}_{n\to\infty} \chi_n(t) = \int_0^1 (1-u)^{1+k}\, d(u^2 \cos tu) = -d^2z/dt^2,$$

where $$z = \Gamma(k+2)\, t^{-k-1} C_{k+1}(t).$$

Hence by the same process as in the preceding article,

$$\operatorname*{Lt}_{n\to\infty} I_2 = \beta \int_0^B t\lambda(t) = \beta \Big[t\, dz/dt - z \Big]_0^B$$

$$= \beta\Gamma(k+2)\Big[t^{-k-1}C_{k+1}(t) - t\frac{d}{dt} t^{-k-1}C_{k+1}(t) \Big]_0^B$$

$$= \beta\Gamma(k+2)\Big[(k+2)\, t^{-k-1}C_{k+1}(t) - t^{-k}C_k(t) \Big]_0^B.$$

Since B is as large as we please, the part due to the upper limit of integration may be neglected, and we get

$$\operatorname*{Lt}_{n\to\infty} I_2 = \beta\Gamma(k+2)\,[(k+2)/\Gamma(k+2) - 1/\Gamma(k+1)] = \beta.$$

This proves the theorem.

§ 10. The companion theorem is the following :—

THEOREM 4.—*If b_n is the typical coefficient of the Fourier series of an odd function which is such that*

(i) $x^{-2}\int_0^x \big| d[xf(x)] \big|$ *is a bounded function of x in the whole interval of periodicity ;*

(ii) $2x^{-2}\int_0^x f(v)\, dv$ *has a unique limit β as $x \to 0$;*

then $(C, 1+k)\,(n^2 a_n) \to 0$, *as* $n \to \infty$.

The proof is the same as that of the preceding article, cosines being replaced by sines.

The part I_1 of the integral is, as before, negligible. In the calculation of I_2, the part due to the interval $(0, B/n)$, the argument proceeds as before till we come to the calculation of the function $\lambda(t)$. We then have

$$\lambda(t) = \int_0^1 (1-u)^{1+k}\, d(u \sin tu)$$

$$= -\frac{d^2}{dt^2}(1+k)\,\Gamma(k+1)\, t^{-k-1} C_{k+2}(t),$$

whence, by the same reasoning as before, our present result follows, since $t^{-k-1}C_{k+2}(t) = 0$, when $t = 0$.

§ 11. The following theorem might be proved precisely in the same way as those which precede it. The proof here given is slightly different, and might equally well have been employed in the other cases.

THEOREM 5.—*If* a_n *is the typical coefficient of the Fourier series of the even function* $f(x)$, *which is such that*

(i) $x^{-3}\int_0^x \left| d\,[xf(x)] \right|$ *is bounded, and*

(ii) $3\,!\,x^{-3}\int_0^x f(x)\,dx$ *has a unique limit* β *as* $x \to 0$;

then $(\mathrm{C},\ 2+k)\,(n^3 a_n) \to 0$, *as* $n \to \infty$.

We have, as before,

$$\frac{\pi}{2}(\mathrm{C},\ 2+k)\,(n^3 a_n)$$

$$= \int_0^\pi \sum_{r=1}^{n-1}\Big[r^3 \cos rv - (r-1)^3 \cos (r-1)\,v\Big]\,(1-r/n)^{k+2} f(v)\,dv$$

$$= \int_0^\pi \psi'_n(v) f(v),$$

where
$$\psi_n(v) = \sum_{r=1}^{n-1}\Big[r^2 \sin rv - (r-1)^2 \sin (r-1)\,v\Big](1-r/n)^{k+2}.$$

Thus $n^k v^{2+k}\psi_n(v)$ is bounded for all positive values of v. We divide the integral, as before, into two parts, I_1 and I_2, where, integrating by parts,

$$\mathrm{I}_1 = \int_{\mathrm{B}/n}^\pi \psi'(v) f(v)\,dv = \Big[n^k v^{2+k}\psi_n\, v\ .\ v^{-2} f(x)\ .\ n^{-k} v^{-k}\Big]_{\mathrm{B}/n}^\pi - \int_{\mathrm{B}/n}^\pi \psi_n(v)\,df(v).$$

Here the quantity in square brackets may be neglected, for by (i) $v^{-2}f(v)$ is bounded, and, by what has already been remarked, $n^k v^{2+k}\psi_n(v)$ is bounded, so that the last factor, $n^{-k}v^{-k}$, determines the value, and this is as small as we please when n and B are taken sufficiently large. Also,

$$\int_e^\pi \psi_n(v)\,df(v) = n^{-k}\int_e^\pi n^k \psi_n(v)\ .\ df(v) \to 0,\ \text{as}\ n \to \infty,$$

therefore, writing $g(v)$ for the function of bounded variation $vf(v)$,

$$\mathrm{I}_1 = \int_{\mathrm{B}/n}^e \psi_n(v)\,df(v) = \int_{\mathrm{B}/n}^e \psi_n(v)\,d\,[v^{-1} g(v)]$$

$$= \int_{\mathrm{B}/n}^e \psi_n(v)\,v^{-1}\,dg(v) - q\int_{\mathrm{B}/n}^e \psi_n(v)\,v^{-2} g(v)\,dv,$$

where the latter integral, being $= \int_{\mathrm{B}/n}^e n^k v^{2+k}\psi_n(v)\ .\ v^{-2} f(x)\ .\ n^{-k} v^{-1+k}\,dv$, is numerically less than a certain constant multiple of $\int_{B/n}^{e} n^{-k} d(v^{-k})$, which is again as small as we please.

On the other hand, writing $G(v)$ for $\int_0^v \left| dg(v) \right|$,

$$\left| \int_{B/n}^{e} \psi_n(v) v^{-1} dg(v) \right| \leq \int_{B/n}^{e} \left| n^k v^{k+2} \psi_n(v) \right| (nv)^{-k} v^{-3} \, dG(v) \leq K \Big[(nv)^{-k} \Big]_{B/n}^{e},$$

K being a constant. This is again as small we please. Thus, finally, I_1 is numerically as small as we please, B and n being chosen sufficiently large. Again, changing the variable in calculating I_2, as before, and using $\chi_n(t)$ for $n^{-3}\psi'_n(t/n)$, we see that $t^3 \frac{d}{dt} \chi_n(t)$ is bounded, and has for limiting function, when $n \to \infty$, $t^3 \frac{d}{dt} \lambda(t)$, where

$$\lambda(t) = \underset{n \to \infty}{\mathrm{Lt}} \, \chi_n(t) = \int_0^1 (1-u)^{2+k} d(u^3 \cos tu) = (2+k) \int_0^1 (1-u)^{k+1} u^3 \cos tu \, du$$

$$= -\frac{d^3}{dt^3} (2+k) \int_0^1 (1-u)^{k+1} \sin tu \, du$$

$$= -\frac{d^3}{dt^3} \Gamma(3+k) t^{-k-2} C_{k+3}(t) = -\frac{d^3 z}{dt^3}, \text{ say.}$$

We thus get

$$I_2 = \int_0^{B/n} \psi'_n(v) f(v) \, dv = \int_0^B \chi_n(t) \, n \, dF(t/n)$$

$$= \Big[\chi_n(t) \, n \, F(t/n) \Big]_0^B - \int_0^B n t^{-3} F(t/n) \, . \, t^3 \frac{d}{dt} \chi_n(t) \, dt.$$

Proceeding to the limit, and integrating on the right term by term, we get

$$3 \underset{n \to \infty}{\mathrm{Lt}} \, I_2 = \Big[\lambda(t) t^3 \beta \Big]_0^B - \int_0^B \beta t^3 \frac{d}{dt} \lambda(t) \, dt$$

$$= \Big[\lambda(t) t^3 \beta \Big]_0^B - \int_0^B \frac{d}{dt} \Big[t^3 \lambda(t) - 3t^2 \lambda(t) \Big] dt,$$

whence

$$\underset{n \to \infty}{\mathrm{Lt}} \, I_2 = \tfrac{1}{2}\beta \int_0^B t^2 \lambda(t) \, dt = -\tfrac{1}{2}\beta \int_0^B \frac{d}{dt} \left(t^2 \frac{d^2 z}{dt^2} - 2t \frac{dz}{dt} + 2z \right) dt.$$

The value of $t^2 z'' - 2tz' + 2z$ at the upper limit of integration is as small as we please, provided B be sufficiently large. We are therefore only concerned with its value at the origin, which is easily seen to be zero. This proves the theorem.

§ 12. The companion theorem is the following:—

THEOREM 6.—*If b_n is the typical coefficient of the Fourier series of the odd function $f(x)$, which is such that*

(i) $x^{-3}\int_0^x \left| d\,[x f(x)] \right|$ *is bounded, and*

(ii) $3!\,x^{-3}\int_0^x f(x)\,dx$ *has a unique limit β as $x \to 0$,*

then $(\mathrm{C},\, 2+k)\,(n^3 b_n) \to 2\beta/\pi$, *as* $n \to \infty$.

As in the corresponding earlier theorems, we now only have to change the cosines into sines. There is no alteration of interest till we come to the expression for the function $\lambda(t)$. We then have

$$\lambda(t) = \int_0^1 (1-u)^{2+k}\, d\,(u^3 \sin tu) = (2+k)\int_0^1 (1-u)^{1+k} u^3 \sin tu\, du$$

$$= +\frac{d^3}{dt^2}(2+k)\int_0^1 (1-u)^{k+1} \cos tu\, du = \frac{d^3}{dt^3}\Gamma(3+k)t^{-k-2}\ \mathrm{C}_{k+2}(t) = \frac{d^3 z}{dt^3},\ \text{say.}$$

We thus get

$$\underset{n\to\infty}{\mathrm{Lt}}\ \mathrm{I}_2 = \tfrac{1}{2}\beta\int_0^{\mathrm{B}} t^2\lambda(t)\,dt = \tfrac{1}{2}\beta\Big[t^2 z'' - 2tz' + 2z\Big]_0^{\mathrm{B}}$$

$$= \tfrac{1}{2}\beta\Gamma(3+k)\Big[(k+4)(k+3)\,t^{-k-2}\mathrm{C}_{k+2}(t) - 2(k+3)\,t^{-k-1}\mathrm{C}_{k+1}(t) + t^{-k}\mathrm{C}_k(t)\Big]_0^{\mathrm{B}}.$$

As before we need only retain the part of this expression due to the origin, which is

$$\tfrac{1}{2}\beta\,[(k+4)(k+3) - 2(k+3)\Gamma(k+3)/\Gamma(k+2) + \Gamma(k+3)/\Gamma(k+1)] = \beta.$$

This proves the theorem.

§ 13. The general theorem and its companion theorem are now evident. We may add that these theorems may be deduced from one another by an inductive process. I have purposely adopted the procedure in the text, as the processes employed seem to be sufficiently instructive in themselves to be worthy of publication.